Lecture 7

Second Order Differential Equations

Ch. 2. Differential Equations

1. An ordinary differential equation of the Nth order is a relation of the form

$$F\left(x, u, \frac{du}{dx}, \ldots, \frac{d^N u}{dx^N}\right) = 0 \qquad (1)$$

An equation of this form is called linear ODE of the Nth order

$$t(x)\frac{d^N u}{dx^N} + \cdots + s(x)\frac{du}{dx} + r(x)u + q(x) = 0$$

For the solutions one has to specify the class of functions to which the solution belong. We might ask the solution

- infinitely differentiable
- N times differentiable
- or differentiable in the sense of generalized functions

- One must be carefull about the solution space. If the class of admissible function is too restricted the equation may have no solution at all belonging to this class.

Example $\frac{du}{dx} = |x| = \begin{matrix} x & x \geq 0 \\ -x & x \leq 0 \end{matrix}$

$$u(x) = \begin{cases} \frac{1}{2}x^2 & x \geq 0 \\ -\frac{1}{2}x^2 & x \leq 0 \end{cases}$$

$$\frac{d^2u}{dx^2} = \Theta(x), \quad \frac{d^3u}{dx^3} = \text{distribut.} = \delta(x)$$

Hence the third derivative exists in the sense of distributions

- In this chapter we shall assume that the solutions are N-times differentiable
- In general and ODE has solutions which may not be unique. We have additional conditions on the function $u(x)$ and $\frac{du}{dx}, \ldots, \frac{d^{N-1}u}{dx^{N-1}}$ at some given point $x = x_0$. These conditions are called boundary conditions
- There is one type of boundary condition for which a unique solution of ODE is ensured under rather weak conditions. We shall state the existence theorem

- <u>Lipschitz condition</u>. Let $f(y_1, y_2, \cdots, y_N)$ be a function of N arguements $y_1, y_2, \cdots, y_N$ and suppose that arguement y_i say, varies withing an interval

$$c_i - \eta \leq y_i \leq c_i + \eta \quad \text{or} \quad |y_i - c_i| \leq \eta$$

where η is a positive number and c_i is some constant. Let $y_{(i)}^{(1)}$ and $y_{(i)}^{(2)}$ be two arbitrary points of the interval given above Then if there exist a positive number k such that

$$|f(y_1, y_2, \cdots, y_i^{(2)}, y_{i+1}, \cdots, y_n) - f(y_1, y_2, \cdots, y_i^{(1)}, y_{i+1}, \cdots y_N)|$$

$$\leq k\,|y_i^{(2)} - y_i^{(1)}|$$

the function f is said to obey a "<u>Lipschitz condition</u>" with respect to the arguement y_i in the above integral

We have now an existance theorem due to Cauchy and Lipschitz

We can write any ODE as

$$\frac{d^N u}{dx^N} = H\left(x, u, \frac{du}{dx}, \ldots, \frac{d^{N-1}u}{dx^{N-1}}\right) \qquad 4.1$$

with the boundary condition at some point x_0

$$u(x_0) = c_0, \quad \frac{du}{dx} = c_1, \ldots, \frac{d^{N-1}u}{dx^{N-1}} = c_{N-1} \qquad (4.2)$$

where $c_0, \ldots, c_{N-1}$ are some constants.

Theorem: Consider the differential equation (4.1) and the boundary condition (4.2). If the function H in (4.1) is continuous and if there exists a positive number M such that, whenever $a \le x \le b$, H obeys Lipschitz condition with respect to its arguement $u, \frac{du}{dx}, \ldots, \frac{d^{N-1}u}{dx^{N-1}}$, when these arguements vary within the

$$\begin{aligned} c_0 - M &\le u \le c_0 + M \\ c_1 - M &\le \frac{du}{dx} \le c_1 + M \\ &\cdots \\ c_{N-1} - M &\le \frac{d^{N-1}u}{dx^{N-1}} \le c_{N-1} - M \end{aligned} \qquad (4.3)$$

a solution of (4.1) satisfying (4.2) exists at all points of the interval and is unique

Boundary conditions are in general given as follows: For a linear ODE of the 2nd order

$$A(x)\frac{d^2u}{dx^2} + B(x)\frac{du}{dx} + C(x)\,u = f(x), \qquad (5.1)$$

in the interval $[a,b]$ and the BCs are

$$\alpha_1\, u(a) + \beta_1 \left.\frac{du}{dx}\right|_{x=a} + \gamma_1\, u(b) + \delta_1 \left.\frac{du}{dx}\right|_{x=b} = \sigma_1$$

$$\alpha_2\, u(a) + \beta_2 \left.\frac{du}{dx}\right|_{x=a} + \gamma_2\, u(b) + \delta_2 \left.\frac{du}{dx}\right|_{x=b} = \sigma_2.$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \sigma_1, \sigma_2$ are constants. If $\sigma_1 = \sigma_2 = 0$, then the BCs are called <u>homogeneous</u>.

2. Second order Differential Equations

Consider now the following second order linear, inhomogenous differential equation:

$$a(x)\frac{d^2u}{dx^2} + b(x)\frac{du}{dx} + c(x)u = q(x), \quad (a \neq 0). \qquad (2.1)$$

Associated homogeneous equation $(q=0)$.

$$a(x)\frac{d^2u}{dx^2} + b(x)\frac{du}{dx} + c(x)u(x) = 0 \qquad (2.2)$$

a) Let $u_1(x)$ and $u_2(x)$ be two solutions of Eq (2.2). Because of the linearity of the equation

$c_1 u_1 + c_2 u_2$, c_1 and c_2 are arbitrary constants

is also a solution of (2.2).

We shall show that if u_1 and u_2 are linearly independent solutions Eq. (2.2), then any solution u of Eq. (2.2) can be given as a linear superposition of u_1 and u_2

i.e $\quad u(x) = c_1 u_1(x) + c_2 u_2(x). \qquad (2.3)$

b). If u_1 and u_2 are linearly independent functions in an interval, then the relation

$$\alpha u_1(x) + \beta u_2(x) = 0 \qquad x \in I \qquad (2.4)$$

implies that $\alpha = \beta = 0$.

If we differentiate (2.4) in I we get

$$\alpha u_1' + \beta u_2' = 0 \qquad (2.5)$$

Hence if u_1 and u_2 are linearly independent, it is sufficient that the Wronskian

$$W(u_1, u_2) = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = u_1 u_2' - u_1' u_2 \neq 0 \qquad (2.6)$$

It is easy to show that if $W(u_1, u_2) = 0$ then u_1 and u_2 are linearly dependent.

$$W(u_1, u_2) = u_1 u_2' - u_1' u_2 = 0$$

$$\frac{u_1'}{u_1} - \frac{u_2'}{u_2} = 0 \qquad u_2 = \text{const.}\, u_1$$

Hence u_1 and u_2 are linearly dependent.

If there are three soln of the h.eq.

$$\left.\begin{aligned} a u_1'' + b u_1' + c u_1 &= 0 \\ a u_2'' + b u_2' + c u_2 &= 0 \\ a u_3'' + b u_3' + c u_3 &= 0 \end{aligned}\right\} \quad (12)$$

$$\begin{pmatrix} u_1'' & u_1' & u_1 \\ u_2'' & u_2' & u_2 \\ u_3'' & u_3' & u_3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{vmatrix} u_1'' & u_1' & u_1 \\ u_2'' & u_2' & u_2 \\ u_3'' & u_3' & u_3 \end{vmatrix} = 0$$

for nontrivial a, b, c (not depends) change

$$\begin{vmatrix} u_1'' & u_2'' & u_3'' \\ u_1' & u_2' & u_3' \\ u_1 & u_2 & u_3 \end{vmatrix}$$ transpose

$$\Rightarrow \begin{vmatrix} u_1 & u_2 & u_3 \\ u_1' & u_2' & u_3' \\ u_1'' & u_2'' & u_3'' \end{vmatrix} = 0 \quad (13)$$

change rows

This mean that u_1 and u_2 and u_3 are linealy dependent. Hence the most the most general soln of the h.eq can be given by

$$u = c_1 u_1(x) + c_2 u_2(x) \quad (14)$$

two linealy independent soln of the h.eq. are called the a <u>fundamental set</u> of solutions. Hence we have

Hence we have

Theorem 2. The most general solution of a homogeneous linear ODE of the second order is of the form

$$u(x) = c_1 u_1(x) + c_2 u_2(x) \quad (15)$$

where c_1 and c_2 are arbitrary constants (complex or real) and u_1 and u_2 are a fundamental set of solutions of the h.eq.

i.e. $W(u_1, u_2) \neq 0$ (16)

Theorem 3. The most general solution of the general second order inhomogeneous linear differential equation is

$$u(x) = c_1 u_1(x) + c_2 u_2(x) + v_p(x) \quad (17)$$

where v_p is any particular solution of the inhom. eq., c_1 and c_2 are arbitrary constants and u_1 and u_2 are a fundamental set solution of the homog. eq.

3. Method of variation of constants

a) Given u_1 we can determine u_2

Let $u_2 = h u_1$. Since u_2 satisfies the homogeneous eqn.

$$a u_2'' + b u_2' + c u_2 = 0$$

$$a(h'u_1 + h u_1')' + b(h'u_1 + h u_1') + c h u_1 = 0$$

$$a(h''u_1 + 2h'u_1' + \cancel{h u_1''}) + \cancel{b h u_1'} + b h' u_1 + \cancel{c h u_1} = 0$$

$$h(a u_1'' + b u_1' + c u_1) + (a h'' + b h')u_1 + 2a h' u_1' = 0$$

since u_1 is a solution, then.

$$(a h'' + b h')u_1 + 2a h' u_1' = 0$$

or

$$a \frac{h''}{h'} + b + 2a \frac{u_1'}{u_1} = 0$$

$$\frac{h''}{h'} + 2 \frac{u_1'}{u_1} + \frac{b}{a} = 0$$

$$\left[\ln (h' u_1^2)\right]' = -\frac{b}{a} \Rightarrow$$

$$h' u_1^2 = \text{const.}\, e^{-\int^x \frac{b(x')}{a(x')} dx'}$$

$$h'(x) = \frac{c}{u_1^2}\, e^{-\int^x \frac{b(x')}{a(x')}\,dx'}$$

$$h(x) = c \int^x \frac{dx'}{u_1^2(x')}\, e^{-\int^{x'} \frac{b(\xi)\,d\xi}{a(\xi)}}$$

$$\Rightarrow \quad u_2(x) = c\; u_1(x) \int^x dx' \frac{1}{u_1^2(x')}\, e^{-\int^{x'} \frac{b(\xi)\,d\xi}{a(\xi)}}$$

"Liouville's Formula"

u_1 and u_2 must be linearly independent

$$W(u_1, u_2) = u_1 u_2' - u_1' u_2 = u_1(h' u_1 + \cancel{h u_1'}) - \cancel{u_1' h u_1}$$

$$= h' u_1^2$$

$$= c\, e^{-\int^x \frac{b(x')\,dx'}{a(x')}}$$

hence $c \neq 0$. Then u_1 and u_2 are called the fundamental set of solns of the homogeneous eqn.

b) Given u_1 find the particular solution u_p.

$$a u_p'' + b u_p' + c u_p = q.$$

let $u_p(x) = v(x)\, u_1(x)$. Then.

$$a(v'u_1 + v u_1')' + b(v'u_1 + v u_1') + c v u_1 = q.$$

$$a(v''u_1 + 2v'u_1' + \cancel{v u_1''}) + b(v'u_1 + \cancel{v u_1'})$$
$$+ c\cancel{v u_1} = q.$$

$$v\underbrace{(a u_1'' + b u_1' + c u_1)}_{=0} + a v'' u_1 + 2 a v' u_1' + b v' u_1 = q.$$

$$\Rightarrow \quad v'' + 2v' \frac{u_1'}{u_1} + \frac{b}{a} v' = \frac{q}{a u_1}.$$

$$v'' + \left(2\frac{u_1'}{u_1} + \frac{b}{a}\right) v' = \frac{q(x)}{a\, u_1(x)}.$$

since $\quad 2\dfrac{u_1'}{u_1} + \dfrac{b}{a} = -\dfrac{h''}{h'}$ (from part a)

$$\Rightarrow \quad v'' - \frac{h''}{h'} v' = \frac{q}{a u_1}$$

$$h' \frac{d}{dx}\left[\frac{v'}{h'}\right] = \frac{q}{a u_1}$$

$$\frac{d}{dx}\left(\frac{v'}{h'}\right) = \frac{q}{a\,u_1 h'} = \frac{q(x)\,u_1(x)}{a\,h'\,u_1^2}$$

$$= \frac{q(x)\,u_1(x)}{a\,W(u_1,u_2)}$$

$$\frac{v'}{h'} = \int^{x} \frac{q(x')\,u_1(x')\,dx'}{a(x')\,W(u_1,u_2)}$$

$$v' = h' \int^{x} \frac{q(x')\,u_1(x')\,dx'}{a(x')\,W(u_1,u_2)}$$

$$= \frac{d}{dx}\left[h \int^{x} \frac{q(x')\,u_1(x')\,dx}{a(x')\,W(u_1,u_2)}\right] - h(x)\,\frac{q(x)\,u_1(x)}{a(x)\,W(u_1,u_2)}$$

$$v(x) = h \int^{x} \frac{q(x')\,u_1(x')\,dx'}{a(x')\,W(u_1,u_2)} - \int^{x} \frac{q(x')\,u_2(x')}{a(x')\,W(u_1,u_2)}\,dx'$$

$$u_p = u_2 \int^{x} \frac{q(x')\,u_1(x')\,dx'}{a(x')\,W(u_1,u_2)} - u_1 \int^{x} \frac{q(x')\,u_2(x')}{a(x')\,W(u_1,u_2)}\,dx'$$

Lecture 8

The method of Green's Function

Generalized Green's Identity

a linear differential operator L in a function space $\mathcal{F}$ whose action on a vector $|u\rangle \in \mathcal{F}$ represented by a differential form $L_x u(x)$ where

$$L = a_0 + a_1 \frac{d}{dx} + a_2 \frac{d^2}{dx^2} + \cdots + a_N \frac{d^N}{dx^N} \tag{1}$$

Suppose that there exist a formal operator $L^\dagger$ with the property that the ~~quantity~~ equation

$$w\left[\bar{v}\, L u - u\,\overline{(L^\dagger v)}\right] = \frac{d}{dx} Q(u, \bar{v}) \tag{2}$$

called the "Lagrange Identity" where Q is a function of $u, \bar{v}$ and their derivatives. For a second order differential operator L the surface term Q has the form

$$Q(u,\bar{v}) = a\, u\bar{v} + b \frac{du}{dx}\bar{v} + c\, u \frac{d\bar{v}}{dx} + d \frac{du}{dx}\frac{d\bar{v}}{dx} \tag{3}$$

and $L^\dagger$ is called the formal <u>adjoint operator</u>
RHS of (2) is called the <u>surface term</u>

<u>Example</u> 1) $L = \frac{d}{dx}$, $\quad \int_a^b \bar{v}\, L u\, dx = \int_a^b \bar{v} \frac{d}{dx} u\, dx$

$$= \bar{v} u \Big|_a^b - \int_a^b u \frac{d}{dx} \bar{v}\, dx$$

$$\int_a^b \left[v \frac{d}{dx} u + u \frac{d\bar{v}}{dx} \right] dx = \int_a^b \left[v \frac{d}{dx} u - u\overline{\left(-\frac{d}{dx} v\right)} \right] dx = Q \Big|_a^b$$

$L^\dagger = -\frac{d}{dx}$, $\quad Q = u\bar{v}$

$$\bar{v} L u = \bar{v} \frac{d}{dx} u = \frac{d}{dx}(\bar{v}u) - u \frac{d\bar{v}}{dx}$$

$$\bar{v} L u - (L^\dagger v) u = \frac{d}{dx}(\bar{v}u) -$$

2) $L = i\frac{d}{dx}$, $\int_a^b \bar{v} L u \, dx = \int_a^b \bar{v}\, i \frac{d}{dx} u \, dx$

$$= i\bar{v}u\Big|_a^b - \int_a^b u\left(i\frac{d}{dx}\bar{v}\right) dx$$

$$\Rightarrow \quad \int_a^b \left[\bar{v} L u + u\left(i\frac{d}{dx}\bar{v}\right)\right] dx = \int \left[\bar{v} L u - \bar{u}\,\overline{\left(+i\frac{d}{dx} v\right)}\right] dx$$

$L^\dagger = i\frac{d}{dx}$, $Q = iu\bar{v}$

3) $L = \frac{d^2}{dx^2}$, $L^\dagger = \frac{d^2}{dx^2}$, $Q = \bar{v}\frac{du}{dx} - u\frac{d\bar{v}}{dx}$

$$\bar{v}\frac{d^2}{dx^2}u = \frac{d}{dx}\left(\bar{v}\frac{du}{dx} - u\frac{d\bar{v}}{dx}\right) + u\frac{d^2\bar{v}}{dx^2}$$

$$\bar{v}\frac{d^2}{dx^2}u - u\frac{d^2\bar{v}}{dx^2} = \frac{d}{dx}Q, \qquad Q = \bar{v}\frac{du}{dx} - u\frac{d\bar{v}}{dx}$$

Integrating the Lagrange identity we get

$$\int_a^b w\left[\bar{v} L u - u\,\overline{(L^\dagger v)}\right] dx = Q(u,\bar{v})\Big|_{x=b} - Q(u,\bar{v})\Big|_{x=a} \qquad (4)$$

This is known as the <u>Generalized Green's identity</u>
RHS is called the <u>boundary</u> or <u>surface</u> term

4) $L = a(x)\frac{d^2}{dx^2}$ 5) $L = b(x)\frac{d}{dx}$

Green's Identity and Adjoint Boundary Conditions

Consider now boundary conditions on u (Homogenous BCs)

$$B_1(u) = \alpha_1 u(a) + \beta_1 u'(a) + \gamma_1 u(b) + \delta_1 u'(b) = 0 \qquad (5)$$
$$B_2(u) = \alpha_2 u(a) + \beta_2 u'(a) + \gamma_2 u(b) + \delta_2 u'(b) = 0$$

Assume that the boundary term vanishes

$$Q(u,\bar{v})\Big|_{x=a} - Q(u,\bar{v})\Big|_{x=b} = 0 \qquad (6)$$

We use (5) and (6) to find the BCs for $\bar{v}$ which are called the adjoint boundary conditions

Examples: 1) $L = \frac{d}{dx}$, $Q = u\bar{v}$

$$u(a)\bar{v}(a) - u(b)\bar{v}(b) = 0$$

one BC. if $u(a) = u(b) \Rightarrow v(a) = v(b)$.

In this case the space of funcn $U = V$

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2) $L = i\frac{d}{dx}$ with $u(a) = v(b)$, $Q = iu\bar{v}$

self adjoint opr. $L^{+} = L$ $\qquad U = V$

L is hermitian

if $u(a) = 2u(b) \Rightarrow 2V(a) = V(b)$

$$U \neq V$$

hence $L^{+} \neq L$

When the surface term ⊕ vanishes to find the Adjoint BCs we get the Green's identity

$$\int_0^b w\left[\bar{v}Lu - u\overline{(L^\dagger v)}\right]dx = 0 \qquad \text{"Green's Identity"} \quad (6)$$

Second Order Self adjoint Operator.

$$L = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x) \qquad \mathrm{Re}\, a(x) > 0 \qquad (7)$$

$$\bar{v}Lu = a\bar{v}u'' + b\bar{v}u' + cu\bar{v}$$

$$= (a\bar{v}u')' - u'(a\bar{v})' + (b\bar{v}u)' - u(b\bar{v})' + cu\bar{v}$$

$$= (a\bar{v}u' + b\bar{v}u)' - [u(a\bar{v})']' + u(a\bar{v})'' - u(b\bar{v})' + cu\bar{v}$$

$$\bar{v}Lu - u[(a\bar{v})'' - (b\bar{v})' + c\bar{v}] = \varphi'$$

$$\varphi = a\bar{v}u' + b\bar{v}u - u(a\bar{v})'$$

$$\bar{v}Lu - \overline{[(\bar{a}v)'' - (\bar{b}v) + \bar{c}v]} = \varphi'$$

$$L^\dagger v = \frac{d^2}{dx^2}(\bar{a}v) - \frac{d}{dx}(\bar{b}v) + \bar{c}v$$

$$= \bar{a}\frac{d^2}{dx^2}v + (2\bar{a}' - \bar{b})\frac{dv}{dx} + (\bar{c} - \bar{b}' + \bar{a}'')v \qquad (8)$$

We took $w = 1$, if a, b, c are real and $b = a'$ then

$$L^\dagger = L = a\frac{d^2}{dx^2} + a'\frac{d}{dx} + c$$

$$= \frac{d}{dx}a\frac{d}{dx} + c. \qquad (9)$$

Theorem. let $p(x)$ and $w(x)$ be real then the the operator L defined by

$$Lu = \frac{1}{w}\frac{d}{dx}\left[p\frac{du}{dx}\right] + cu \qquad (10)$$

is self adjoint with respect to a weight function w.

proof:

$$w\bar{v}Lu = \bar{v}\frac{d}{dx}p\frac{du}{dx} + cu\bar{v}$$

$$= \frac{d}{dx}\left(\bar{v}p\frac{du}{dx}\right) - p\frac{d\bar{v}}{dx}\frac{du}{dx} + cu\bar{v}$$

$$= \frac{d}{dx}\left[p(\bar{v}u' - u\bar{v}')\right] + u\frac{d}{dx}\left(p\frac{d}{dx}\bar{v}\right) + cu\bar{v}$$

$$\int_a^b w\left[\bar{v}Lu - u\,\overline{\frac{1}{w}\left[\frac{d}{dx}\left(p\frac{d}{dx}v\right) + cv\right]}\right] = \psi\Big|_a^b$$

$$L^{\dagger} = \frac{1}{w}\frac{d}{dx}p\frac{d}{dx} + c \quad , \quad \psi = p(\bar{v}u' - u\bar{v}') \qquad (11)$$

$$= L$$

It is possible to show that any 2nd differential equation with real coefficients into the form.

$$L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c = a\left(\frac{d^2}{dx^2} + \frac{b}{a}\frac{d}{dx}\right) + c$$

Let $b/a = p'/p$.

$$L = a\left(\frac{d^2}{dx^2} + \frac{p'}{p}\frac{d}{dx}\right) + c = \frac{a}{p}\frac{d}{dx}\left(p\frac{d}{dx}\right) + c$$

$$= \frac{1}{w}\frac{d}{dx}\left(p\frac{d}{dx}\right) + c$$

where $w = p/a \quad \Rightarrow \quad a = p/w$

$$b = a\,p'/p = \frac{p'}{w}$$

Theorem: Every linear, formal differential operator of the second order with real coefficients is self-adjoint, provided the weight function $w(x)$ is chosen as

$$w(x) = \frac{1}{a(x)} e^{\int_{x_0}^{x} dx' \frac{b(x')}{a(x')}} \qquad \text{and } a(x) > 0$$

The surface term Eqn. (11).

$$Q = p(\bar{v}u' - u\bar{v}') = a\omega(\bar{v}u' - u\bar{v}') \qquad (12)$$

Hermitian Operators: $L = L^{+}$ self adjoint but we must compare the domain function spaces.

i) $u(a) = u(b) = 0$ (Dirichlet conditions) $\qquad p\bar{v}(b)u'(b) - p\bar{v}(a)u'(a) = 0$

$\Rightarrow v(a) = v(b) = 0.$

ii) $u'(a) = u'(b) = 0$ (Neumann conditions) $\qquad v'(a) = v'(b) = 0$

(?) iii) $\alpha u(a) - u'(a) = \beta u(b) - u'(b) = 0$ $\qquad (\alpha, \beta \text{ real})$
(general unmixed conditions)

$\Rightarrow \alpha v(a) = v'(a)$
$\beta v(b) = v'(b)$

(?) iv) $u(a) = u(b)$
$u'(a) = u'(b)$ $\qquad \Rightarrow \quad p(b)v(b) = p(a)v(a)$
periodic conditions $\qquad p(b)v'(b) = p(a)v'(a)$

$$p(a)(\bar{v}(a)u'(a) - u(a)\bar{v}'(a)) = p(b)(\bar{v}(b)u'(b) - u(b)\bar{v}'(b))$$

$$\big(p(a)\bar{v}(a) - p(b)\bar{v}(b)\big)u'(b) = u(b)\big(p(a)\bar{v}'(a) - p(b)\bar{v}'(b)\big)$$

$$p(a)\bar{v}(a) = p(b)\bar{v}(b)$$
$$p(a)\bar{v}'(a) = p(b)\bar{v}'(b) \quad \checkmark$$

Green's Functions

Let u satisfy the inhomogeneous differential equation

$$L\,u = f \tag{13}$$

with homogeneous boundary conditions

$$B_1(u) = 0 \quad , \quad B_2(u) = 0 \tag{14}$$

The function v satisfying the adjoint differential equation

$$L^{+} v = h \tag{15}$$

with adjoint boundary conditions

$$\text{ad}\, B_1(u) = 0 \quad , \quad \text{ad}\, B_2(u) = 0 \tag{16}$$

where f and ~~g~~ h are given functions.

Here the idea is to find the inverse of the operator L in (13) so that

$$E L u = E f \qquad , \quad E L = \text{id.}$$

$$\langle v | E L | u \rangle = \langle v | E | f \rangle$$

where v is the function in V

$$u = \langle v | E | f \rangle$$

We do this in the following way

We let

$$L\,u = f \qquad (17)$$
$$L^{\dagger} v = h.$$

and define functions $G(x,x')$ and $g(x,x')$ satisfying

$$L_x\, G(x,y) = \frac{\delta(x-y)}{w(x)} \qquad (18)$$

$$L^{\dagger}_x\, g(x,y) = \frac{\delta(x-y)}{w(x)}. \qquad (19)$$

which are called the Green's functions. Using the pairs (u,v), (u,g), (v,G) and (G,g)

$L: u \in U$ In the Green's identity we obtain.

$L^{\dagger}: v \in V$

i) (u,v)

$$\int_a^b w(\bar{v}\,Lu - u\,\overline{(L^{\dagger}v)})\,dx = \int_a^b w\,\bar{v} f\,dx - \int_a^b w\,u\,\bar{h}\,dx = 0$$

$$\langle v,f\rangle = \langle h,u\rangle \qquad (\langle u,h\rangle = \langle f,v\rangle) \,. \qquad (20)$$

ii) (u,g)

$$\int_a^b w[(\bar{g}\,Lu - u\,\overline{(L^{\dagger}g)}]\,dx = \int_a^b w(x)\,\bar{g}(x,y) f(x)\,dx - u(y) = 0$$

$$u(y) = \int_a^b w(x)\,\bar{g}(x,y)\,f(x)\,dx \qquad (21)$$

iii) (v,G)

$$\int_a^b w[(\bar{v}\,LG - G\,\overline{(L^{\dagger}v)}]\,dx = \int_a^b w\,\bar{v}\,\frac{\delta(x-y)}{w}\,dx - \int_a^b w\,G\,\bar{h}$$

$$\bar{v}(y) = \int_a^b w(x)\,\bar{h}(x)\,G(x,y)\,dx$$
$$v(y) = \int_a^b w(x)\,h(x)\,\bar{G}(x,y)\,dx \qquad (22)$$

$G(x,y)\,,\; g(x,z)$ $\qquad$ $L\,G \doteq \dfrac{\delta(x-y)}{w}$

$\qquad\qquad\qquad\qquad\quad$ $L\,g = \dfrac{\delta(x-z)}{w}$

iv) (G, g)

$$\int_a^b w\left[\bar{G}\,L g - g\,(\overline{L^{\dagger} G})\right]dx = \int_a^b w(x)\,\bar{G}(x,y)\,\frac{\delta(x-z)}{w}\,dx - \int_a^b w(x)\,g(x,z)\,\frac{\delta(x-y)}{w}\,dx$$

$$\bar{G}(z,y) = g(y,z) \qquad (23)$$

In solving Green's functions

$$L_x G(x,y) = \frac{\delta(x-y)}{w(x)} \qquad B_1(u)=0,\; B_2(u)=0 \qquad (24)$$

$$L_x^{\dagger}\, g(x,y) = \frac{\delta(x-y)}{w(x)} \qquad ad\,B_1(u)=0,\; ad\,B_2(u)=0 \qquad (25)$$

if L is Hermitian $\left(L = L^{\dagger} \qquad ad\,B_1 = B_1\,,\; ad\,B_2 = B_2\right)$

$$G(x,y) = g(x,y) \qquad (26)$$

$$\Rightarrow \qquad \bar{G}(z,y) = G(y,z) \qquad (27)$$

if L_x is real then

$$G(x,y) = G(y,x) \qquad (28)$$

$$\Rightarrow \quad u(x) = \int_a^b w(y)\,f(y)\,G(x,y)\,dx \qquad (29)$$

$$L_x G(x,y) = \frac{\delta(x-y)}{w} \qquad (30)$$

The method of Green's Function:

Let

$$L_x\, u(x) = f(x), \quad B_i(u) = 0, \quad i = 1,2,\dots,N \qquad (1)$$

be a differential equation with some homogeneous boundary conditions. The adjoint equations and adjoint boundary conditions.

$$L_x^{\dagger}\, v(x) = h(x), \quad \text{and } B_i(v) = 0, \quad i = 1,2,\dots,N \qquad (2)$$

Green's function satisfy also similar equations let $G(x,y)$, $x,y \in I$ be a function of two variables satisfying

$$L_x\, G(x,y) = \frac{\delta(x-y)}{w(x)} \qquad (3)$$

and $g(x,y)$, $x,y \in I$ be a function of two variable satisfying

$$L_x^{\dagger}\, g(x,y) = \frac{\delta(x-y)}{w} \qquad (4)$$

These functions are called the corresponding Green's functions.

Let us recall the Green's identity (surface term $= 0$)

$$\int_I w(x)\left[\overline{\psi} L_x \phi - \phi\, \overline{(L_x^{\dagger}\psi)}\right] dx = 0 \qquad (5)$$

i) Let $\phi = u(x)$, $\psi = g(x,y)$

$$u(y) = \int_I w(x)\, f(x)\, \overline{g}(x,y)\, dx \qquad (6)$$

ii) $\phi(x) = v(x)$, $\psi(x) = G(x,y)$

$$v(y) = \int_I w(x)\, h(x)\, \bar{G}(x,y)\, dx \qquad (7)$$

iii) $\phi(x) = u$, $\psi = v$

$$\langle v, f \rangle = \langle h, u \rangle \qquad (8)$$

iv) $\phi(x) = G(x,y)$, $\psi = g(x,z)$

$$\int \bar{g}(x,z)\, \delta(x-y)\, dx = \int G(x,y)\, \delta(x-z)\, dx$$

$$\bar{g}(y,z) = G(z,y) \qquad (9)$$

Remarks:

- If L_x is self adjoint $g(x,y) = G(x,y)$
  Hence

$$G(x,y) = \bar{G}(y,x) \qquad (10)$$

- If the DE is ~~a~~ real DE then

$$G(x,y) = G(y,x) \qquad (11)$$

symmetric Green's function

Properties of the Green's functions

The differential equation satisfied by Green's function

$$L_x\, G(x,y) = a(x)\frac{\partial^2}{\partial x^2} G + b(x)\frac{\partial}{\partial x} + c(x)\, G = \frac{\delta(x-y)}{w(x)} \qquad (12)$$

for $x \in I$ except $x = y$.

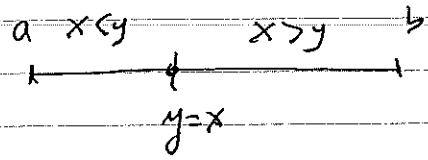
We assume $a(x) > 0$ $x \in I$. Defining a function $p(x)$ so that

$$\frac{p'}{p} = \frac{b}{a} \Rightarrow p = e^{\int^x d\xi \frac{b(\xi)}{a(\xi)}}$$

$$\Rightarrow \quad \frac{a(x)}{p(x)}\frac{\partial}{\partial x}\left[p(x)\frac{\partial G}{\partial x}\right] + c(x)\, G = \frac{\delta(x-y)}{w(x)}.$$

or

$$\frac{\partial}{\partial x}\left[p(x)\frac{\partial G}{\partial x}\right] + \frac{p(x)\,c(x)}{a(x)}\, G = \frac{p(x)\,\delta(x-y)}{a(x)\,w(x)} \qquad (12)$$



$G(x,y)$ is a continuous function everywhere in I. This means it is also cont. at $x=y$

$$G(x,y)\Big|_{x=y-\varepsilon} = G(x,y)\Big|_{x=y+\varepsilon}$$

As $\varepsilon \to 0$ G

$$G_<(x,y) = G_>(x,y). \qquad (13)$$

Because of (11) $G(x,y)$ may not have continuous derivative at $x=y$.

Lemma:

$$p(y)\left[G'_> - G'_<\right] = \frac{1}{a\,w(y)}$$

proof: integrate (12) over $[y-\varepsilon,\ y+\varepsilon]$

$$p(x)\frac{\partial G}{\partial x}\Big|_{y-\varepsilon}^{y+\varepsilon} + \int_{y-\varepsilon}^{y+\varepsilon} \frac{p(x)\,c(x)}{a(x)}\,G(x,y)\,dx$$

$$= \int_{y-\varepsilon}^{y+\varepsilon} \frac{p(x)\,\delta(x-y)}{a(x)\,w(x)}\,dx$$

$$= \int_{-\varepsilon}^{\varepsilon} \frac{p(u+y)\,du}{a(u+y)\,w(u+y)}\,\delta(u)$$

$$= \frac{p(y)}{a(y)\,w(y)}$$

$$p(x)\frac{\partial G}{\partial x}\Big|_{y-\varepsilon}^{y+\varepsilon} + \int_{y-\varepsilon}^{y+\varepsilon} \frac{p(x)\,c(x)}{a(x)}\,G(x,y)\,dx = \frac{p(y)}{a(y)\,w(y)}.$$

as $\varepsilon \to 0$ second term goes to zero

$$p(y)\left[G'_x(x,y)\Big|_{x=y_+} - G_x(x,y)\Big|_{x=y_-}\right] = \frac{p(y)}{a(y)\,w(y)}$$

$$G_x(x,y)\Big|_{x=y_+} - G_x(x,y)\Big|_{x=y_-} = \frac{1}{a(y)\,w(y)}$$

Hence we have

$$G(x,y)\Big|_{x=y_-} = G(x,y)\Big|_{x=y_+} \qquad (15)$$

$$G_x(x,y)\Big|_{x=y_+} - G_x(x,y)\Big|_{x=y_-} = \frac{1}{a(y)w(y)} \qquad (16)$$

In addition to these G satisfies the BCs as well

$$B_1(G)=0, \quad B_2(G)=0 \qquad (17)$$

Construction of Green's function
(uniqueness of Green's function)

$G(x,y)$ satisfies the homogeneous eqn for $x \neq y$. hence

$$G(x,y) = \begin{cases} u_< & a \le x < y \\ u_> & y < x \le b. \end{cases} \qquad (18)$$

Where

$$\begin{aligned} u_< &= c_1 u_1(x) + c_2 u_2(x) \\ u_> &= d_1 u_1(x) + d_2 u_2(x). \end{aligned} \qquad (19)$$

where c_1, c_2, d_1, and d_2 are constants (wrt x). to be determined by the use of (15-17).

Using the conditions (15) and (16) we obtain

$$\begin{aligned} &(c_1-d_1)\,u_1(y) + (c_2-d_2)\,u_2(y) \\ &(c_1-d_1)\frac{d}{dy}u_1(y) + (c_2-d_2)\frac{d}{dy}u_2(y) = -\frac{1}{a(y)w(y)}. \end{aligned} \qquad (20)$$

letting $c_1 - d_1 = c$, $c_2 - d_2 = d$

$$c\, u_1(y) + d\, u_2(y) = 0$$

$$c\, u_1'(y) + d\, u_2'(y) = -\frac{1}{a w}$$

$$c = \frac{\begin{vmatrix} 0 & u_2 \\ -\frac{1}{aw} & u_2' \end{vmatrix}}{W} = \frac{u_2(y)}{a(y)\, w(y)\, \overline{W}(u_1, u_2)} \qquad (21)$$

$$d = \frac{\begin{vmatrix} u_1 & 0 \\ u_1' & -\frac{1}{aw} \end{vmatrix}}{W} = -\frac{u_1(y)}{a(y)\, w(y)\, \overline{W}(u_1, u_2)} \qquad (22)$$

Hence
$$c_1 = c + d_1$$
$$c_2 = d + d_2$$

$$G(x,y) = G_1(x,y) + d_1 u_1(x) + d_2 u_2(x) \qquad (23)$$

where
$$G_1(x,y) = \begin{cases} c(y)\, u_1(x) + d(y)\, u_2(y) \\ 0 \end{cases}$$

$$B_1(G) = B_1(G_1) + d_1 B_1(u_1) + d_2 B_1(u_2) = 0 \qquad (24)$$
$$B_2(G) = B_2(G_1) + d_1 B_2(u_1) + d_2 B_2(u_2) = 0 \qquad (25)$$

To have a consistent solution we must get

$$\left\| \begin{matrix} B_1(u_1) & B_1(u_2) \\ B_2(u_1) & B_2(u_2) \end{matrix} \right\| \neq 0 \qquad (26)$$

(9.5)'

$$d_1 B_1(u_1) + d_2 B_1(u_2) = -B_1(G_1)$$
$$d_1 B_2(u_1) + d_2 B_2(u_2) = -B_2(G_1)$$

$$d_1 = \frac{\begin{vmatrix} -B_1(G_1) & B_1(u_2) \\ -B_2(G_1) & B_2(u_2) \end{vmatrix}}{\Delta}$$

$$d_2 = \frac{\begin{vmatrix} B_1(u_1) & -B_1(G_1) \\ B_2(u_1) & -B_2(G_1) \end{vmatrix}}{\Delta}$$

$$\Delta = B_1(u_1) B_2(u_2) - B_1(u_2) B_2(u_1) \neq 0$$

if $\Delta = 0$

$$B_1 (B_2(u_2) u_1 - B_2(u_1) u_2) = 0$$
$$\Rightarrow B_2 (\qquad) = 0$$

Then we obtain a unique Green's function.

If det = 0 we get

$$\Delta = B_1(u_1) B_2(u_2) - B_1(u_2) B_2(u_1) = 0 \qquad (27)$$

$$\Delta = B_1\left(B_2(u_2) u_1 - B_2(u_1) u_2 \right) = 0 \qquad (28)$$

It means that $\quad \alpha = B_2(u_2)$, $\beta = B_2(u_1)$

$$\alpha B_1(u_1) - \beta B_1(u_2) = 0$$
$$\alpha B_2(u_1) - \beta B_2(u_2) = 0 \qquad (29)$$

$B_1(v) = 0$

$v_1 = B_2(u_2) u_1 - B_2(u_1) u_2$

$\Downarrow$

$B_2(v) = 0$ identical

$\exists$ nontrivial soln if $\Delta = 0$. Or

$$B_1(\alpha u_1 - \beta u_2) = 0, \quad B_2(\alpha u_1 - \beta u_2) = 0 \qquad (30)$$

This means that there exist a solution of homogeneous equation satisfying both boundary conditions (NONTRIVAL SOLUTIONS)

<u>Theorem</u>: Let $L_x u = f(x)$ with the homogeneous boundary conditions $B_1(u) = B_2(u) = 0$. provided the homogeneous equation $L_x u = 0$ has no nontrivial solutions satisfying the boundary conditions, the Green's function associated with $L_x u = f$ exist and is unique. The solution $u(x)$ is given by

$$u(x) = \int_a^b dy\, w(y)\, G(x,y)\, f(y) \qquad (31)$$

is unique.

Example: $\frac{d^2u}{dx^2} = f(x)$, $u(0) = u(a) = 0$, $0 \le x \le a$

$$\frac{d^2}{dx^2} G(x,y) = \delta(x-y) \qquad w=1, \quad a=1. \qquad (32)$$

$$u_1 = 1, \quad u_2 = x \qquad (33)$$

$$G(x,y) = \begin{cases} c_1 + d_1 x \\ c_2 + d_2 x \end{cases} \qquad (34)$$

$$c_1 + d_1 y = c_2 + d_2 y$$

$$d_2 - d_1 = 1.$$

$$c_1 - c_2 + (d_1 - d_2) y = 0$$

$$d_1 - d_2 = -1.$$

$$c_1 - c_2 = y.$$

$c_1 = 0$

$c_2 + d_2 a = 0$

$$d_1 = -1 + d_2, \quad c_1 = c_2 + y$$

$$G(x,y) = \begin{cases} c_2 + d_2 x \cancel{+ y} - x \\ c_2 + d_2 x \end{cases} \qquad (35)$$

$$= c_2 + d_2 x + G_1(x,y),$$

$$G_1(x,y) = \begin{cases} y - x \\ 0 \end{cases} \qquad (36)$$

$$B_1(G_1) + c_2 + d_2(0) = 0 \quad \Rightarrow \quad y + c_2 = 0$$

$$B_2(G_1) + c_2 + d_2 a = 0$$

$$\underset{0}{\cancel{ya}} - y + d_2 a = 0 \quad \Rightarrow \quad d_2 = y/a$$

9.8

$$G(x,y) = \begin{cases} \frac{y}{a}x - x = \frac{x}{a}(y-a) \\ -y + \frac{y}{a}x := \frac{y}{a}(x-a) \end{cases} \quad (37)$$

$$u(x) = \int_0^a G(x,y)\, f(y)\, dy \quad (38)$$

$$= \int_0^x \underset{y<x}{G_>}\, f\, dy + \int_x^a \underset{y>x}{G_<}\, f\, dy$$

$$= \int_0^x \frac{y}{a}(x-a) f(y)\, dy + \int_x^a \frac{x}{a}(y-a) f(y)\, dy$$

$$u(x) = \frac{x-a}{a}\int_0^x y\, f(y)\, dy + \frac{x}{a}\int_x^a (y-a) f(y)\, dy \quad (39)$$

Example 2.

$$\frac{d}{dx}\left(p\frac{du}{dx}\right) + wcu = f \qquad a \le x \le b \tag{40}$$

$$B_1(u) = u(a) - \alpha\frac{du}{dx} = 0$$
$$B_2(u) = u(b) - \beta\frac{du}{dx}(b) = 0 \tag{41}$$

$$u_< = c_1 u_1 + c_2 u_2$$
$$u_> = d_1 u_1 + d_2 u_2 \tag{42}$$

$$B_1(u_<) = 0 \Rightarrow c_1 u_1(a) + c_2 u_2(a) = \alpha\left(c_1 u_1'(a) + c_2 u_2'(a)\right)$$
$$B_2(u_>) = 0 \Rightarrow d_1 u_1(b) + d_2 u_2(b) = \beta\left(d_1 u_1'(b) + d_2 u_2'(b)\right) \tag{43}$$

or

$$c_1 B_1(u_1) + c_2 B_1(u_2) = 0$$
$$d_1 B_2(u_1) + d_2 B_2(u_2) = 0$$

$$c_2 = -c_1\frac{B_1(u_1)}{B_1(u_2)}$$

$$d_2 = -d_1\frac{B_2(u_1)}{B_2(u_2)}$$

$$u_< = c_1\left(u_1 - \frac{B_1(u_1)}{B_1(u_2)}u_2\right) \qquad B_1(u_<) = 0$$

$$u_> = d_1\left(u_1 - \frac{B_2(u_1)}{B_2(u_2)}u_2\right) \qquad B_2(u_>) = 0$$

$$u_<(x) = c_1\,\bar{u}_<(x), \qquad \bar{u}_<(x) = u_1(x) - \frac{B_1(u_1)}{B_1(u_2)}u_2$$
$$u_>(x) = d_1\,\bar{u}_>(x) \qquad \bar{u}_>(x) = u_1(x) - \frac{B_2(u_1)}{B_2(u_2)}u_2 \tag{44}$$

$$u_< = u_> \qquad x = y.$$

$$c_1 \left(u_1 - \frac{B_1(u_1)}{B_1(u_2)} u_2 \right) = d_1 \left(u_1 - \frac{B_2(u_1)}{B_2(u_2)} u_2 \right)$$

$$d_1 u_>' - d_2 u_<' = \frac{1}{p(y)}$$

$$c_1 = \frac{\bar{u}_>(y)}{p(y)\,\bar{W}(u_<, u_>)}, \quad d_1 = \frac{\bar{u}_<(y)}{p(y)\,\bar{W}(u_<, u_>)}$$

$$G(x,y) = \begin{cases} \dfrac{\bar{u}_<(x)\,\bar{u}_>(y)}{p(y)\,\bar{W}} & a \le x < y \\ \dfrac{\bar{u}_<(y)\,\bar{u}_>(x)}{p(y)\,\bar{W}} & y < x \le b \end{cases} \qquad (45)$$

Both $\bar{u}_>$ and $\bar{u}_<$ obey the hom. eqns hence

$$\frac{d}{dx} p\bar{W} = 0 \qquad p\bar{W} = \text{constant}$$

$$p(x)\,\bar{W}(x) = p(y)\,\bar{W}(y) = c \ne 0$$

if $c = 0$ $\quad \bar{u}_<(x) = \gamma\, \bar{u}_>(y)$

$$B_1(\bar{u}_<) = 0 \Rightarrow B_1(\bar{u}_>) = 0$$

but $B_2(\bar{u}_>) = 0 \quad B_1(\bar{u}_>) = 0 \qquad (46)$

Nontrivial solution of hom. eqn. satisfies both BCs.

$B_1(u_<) = 0$ and

$B_2(u_<) = 0$

properties of the Green's function

$$L_x\, G(x,y) = a(x)\frac{\partial^2}{\partial x^2} G(x,y) + b(x)\frac{\partial}{\partial x} G(x,y) + c(x)\, G(x,y)$$

$$= \frac{\delta(x-y)}{w} \qquad (31)$$

Let $\qquad p'/p = \dfrac{b}{a} \qquad \Rightarrow p(x) = e^{\int_{x_0}^{x} \frac{b(x')dx'}{a(x')}} \qquad (32)$

$w = p/a$

$\Rightarrow$

$$\frac{a(x)}{p(x)}\frac{\partial}{\partial x}\left(p\frac{\partial G}{\partial x}\right) + c\,G = \frac{\delta(x-y)}{w(x)} \qquad (33)$$

$$\Rightarrow \frac{\partial}{\partial x}\left(p\frac{\partial G}{\partial x}\right) + \frac{cp}{a}\,G = \frac{p(x)\,\delta(x-y)}{a(x)\,w(x)} \qquad (34)$$

let $\quad a \le x, y \le b \quad$ We assume that $G(x,y)$ is continuous everywhere in $[a,b]$. Hence G is continuous at $x=y$

$$G(x,y)\Big|_{x=y_-} = G(x,y)\Big|_{x=y_+} \qquad (35)$$

Integrate (34) for $(y-\varepsilon, y+\varepsilon)$ where $\varepsilon > 0$

$$p\frac{\partial G}{\partial x}\Bigg|_{x=y-\varepsilon}^{x=y+\varepsilon} + \int_{y-\varepsilon}^{y+\varepsilon} \frac{cp}{a}\, G(x,y)\,dx = \frac{p(y)}{a(y)w(y)}$$

let $\varepsilon \to 0$ then

$$p\left(\frac{\partial G}{\partial x}\Big|_{x=y_+} - \frac{\partial G}{\partial x}\Big|_{x=y_-}\right) = \frac{p(y)}{a(y)w(y)} \qquad (36)$$

Then we get

$$G(x,y)\Big|_{x=y_-} = G(x,y)\Big|_{x=y_+} \qquad (37)$$

$$\frac{\partial}{\partial x} G(x,y)\Big|_{x=y_+} - \frac{\partial}{\partial x} G(x,y)\Big|_{x=y_-} = \frac{1}{a(y)\,w(y)} \qquad (38)$$

In addition to these conditions we have also

$$B_1(G)=0 \quad , \quad B_2(G)=0 \qquad (39)$$

Then we have the following theorem

<u>Theorem</u>: Consider the boundary value problem

$L_x u = f(x)$, $x \in (a,b)$ with the boundary conditions $B_1(u)=0$ and $B_2(u)=0$. Here we assume that L_x is a second order order differential ~~equation~~ operator. Provided that that the homogenous equation has no nontrivial solution satisfying the above boundary conditions the Green's function associated with the boundary value problem exists and unique. The solution is given by

$$u(x) = \int_a^b w(y)\, G(x,y)\, f(y)\, dy \qquad (37)$$

proof:

$$G(x,y) = \begin{cases} c_1 u_1(x) + c_2 u_2(x) & a \le x < y \\ d_1 u_1(x) + c_2 u_2(x) & y < x \le b. \end{cases} \qquad (38)$$

where u_1 and u_2 are fundamental solns of $L_x u = 0$.
c_1, c_2, d_1, d_2 are to be determined

We have four unknown function c_1, c_2, d_1 and d_2 (of y) an four conditions given in (37), (38) and the BCs in (39). This proves the existence for the uniqueness we continue to solve $G(x,y)$ in (38).

If there exist nontrivial solutions of hom. eqn satisfyin both boundary conditions.

A) $L\cancel{\varphi} = 0$ $\quad B_1(\cancel{\varphi}) = 0, \; B_2(\cancel{\varphi}) = 0$

$(\cancel{\varphi}, g)$. Green's identity

$$\int_a^b \omega \left[\bar{g}(x,y) \cancel{L_x \varphi}^{\,0} - \varphi \left(\overline{L_x^{\dagger} g} \right) \right] dx = 0$$

$$- \int_a^b \cancel{\omega} \chi \frac{\delta(x-y)}{\cancel{\omega}} \, dx = -\chi(y) = 0$$

contradiction

g is not a correct Green's funct.

$(\chi, v) \rightarrow \langle \chi, h \rangle = 0$
constraint

B) $L_x^{\dagger} \mu = 0$ $\quad ad B_1(\mu) = 0, \; ad B_2(\mu) = 0$

$\Rightarrow (G, \mu) \rightarrow \mu = 0$ $\quad G$ is not a correct Green's fn

$(u, \mu) \rightarrow \langle f, \mu \rangle = 0$
constraint

If there exists nontrivial solns v_i, $i \leq 1,2$ of

$$L^+ v = 0 \qquad \text{with ad } B_i(v) = 0.$$

$$\Rightarrow \langle v, f \rangle = \langle \bar{h}, u \rangle = 0 \qquad (h = 0)$$

$$\text{or} \qquad v(x) = \int h \omega G \, dx = 0 \qquad (47)$$

contradicts that there exists a nontrivial solutn of $L^+ v = h$.

$$L^+ v = 0 \quad \text{with} \quad \text{ad } B_i(v) = 0 \quad \Rightarrow \langle v_i, f \rangle = 0$$

$$L u = 0 \quad \text{with} \quad BCs \quad \Rightarrow \langle \bar{h}, u_i \rangle = 0$$

necessary conditions for the existence of the solutions.

- $L_x u = f \qquad B_i(u) = 0$ (48)

Green's function of this eqn. $L^+ g = \frac{\delta}{\omega}$.
It has homogeneous eqn $L^+ \bar{v} = 0$ if
if There exist a $\bar{v}$ satisfy both ad $B_i(\bar{v}) = 0$ $(h=0)$
To have a soln

$$\langle \bar{v}, f \rangle = 0 \qquad (49)$$

- Similarly $L_x^+ v = h$, ad $B_i(u) = 0$ (50)

Green's funcn of this eqn $L^* G = \frac{\delta}{\omega}$.
It has cont. eqn $L \bar{u} = 0$, $f = 0$
if There exist a $\bar{u}$ satisfy both $B_i(\bar{u}) = 0$ i.e.
To have a soln.

$$\langle h, \bar{u} \rangle = 0 \qquad \text{"consistency"} \qquad (51)$$

Examples of Green's Functions:

1. $L(u) = u''$, $u(0) = u(1) = 0$

a) $x \in (0,1)$

$$G(x,y) = \begin{cases} (1-y)\,x & \text{for } x \le y \\ (1-x)\,y & \text{for } x \geq y \end{cases}$$

b) $u(0) = 0$, $u'(1) = 0$

$$G(x,y) = \begin{cases} x & \text{for } x \le y \\ y & \text{for } x \geq y \end{cases}$$

c) For the interval $-1 \le x \le 1$

BCs $u(-1) = u(1) = 0$

$$G(x,y) = -\frac{1}{2}\left(|x-y| + xy - 1\right)$$

d) $0 \le x \le 1$

$u(0) = -u(1)$, $u'(0) = -u'(1)$

$$G(x,y) = -\frac{1}{2}|x-y| + \frac{1}{4}$$

2. $L(u) = x u'' + u'$

Associated with the Bessel function of zero. th order $J_0(x)$ for the interval $0 \le x \le 1$ and the boundary conditions $u(1) = 0$, $u(0)$ is finite. $\Rightarrow$

$$G(x,y) = \begin{cases} -\log y & x \le y \\ -\log x & x \geq y \end{cases}$$

3. $L(u) = (x u')' - \frac{n^2}{x} u$

$u(1) = 0, \quad u(0)$ finite

associated with the Bessel function $J_n(x)$

$$G(x,y) = \frac{1}{n}\left[\left(\frac{x}{y}\right)^n - (xy)^n \right] \qquad x \le y$$

$$= \frac{1}{n}\left[\left(\frac{y}{x}\right)^n - (xy)^n \right] \qquad x > y$$

4) $L(u) = [(1-x^2)u']' - \frac{h^2}{1-x^2} u$

for $h = 0, 1, 2, \dots$. This equation is associated with the Legendre functions of zero-th order, first order, etc, respectively.

$-1 \le x \le 1$, and the ~~boundary condition~~ solutions of $L(u) = 0$ which are finite at $x=-1$ and $x=1$ they are

$$u_1 = c_1 \left[\frac{(1+x)}{(1-x)} \right]^{h/2}, \quad u_2 = c_2 \left[\frac{1-x}{1+x} \right]^{h/2}$$

with these solutions the GF

(Boundary condition are: u finite at both end points)

$$G(x,y) = \frac{1}{2h}\left(\frac{1+x}{1-x} \quad \frac{1-y}{1+y}\right)^{h/2} \qquad x \le y$$

$$= \frac{1}{2h}\left(\frac{1+y}{1-y} \cdot \frac{1-x}{1+x}\right)^{h/2} \qquad x \ge y$$

This construction fails when $h = 0$

$\Rightarrow L[u] = [(1-x^2)u']' = 0$ has solution (normalised)

$u_1 = \frac{1}{\sqrt{2}}$ which satisfies BCs

Hence we use generalised GF method.

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Generalized Green's function:

i) $L_x u = f \quad , \quad B_i(u) = 0$ (52)

Green's function of this equation

$$L^{+} g(x,y) = \frac{\delta(x-y)}{\omega} \quad (53)$$

If there exist a nontrivial soln of homogeneous eqn. $L^{+}\bar{v} = 0$ satisfying both adj. BCs. To have a solution a necessary condition

$$\langle \bar{v}_i, f \rangle = 0 \qquad v_i \,,\; i \leq 2. \quad (54)$$

ii) $L_x^{+} v = h \quad , \quad \text{adj } B_i(u) = 0$

Green's function of this eqn.

$$L_x G(x,y) = \frac{\delta(x-y)}{\omega} \quad (55)$$

If there exist a nontrivial soln. of homogeneous eqn $L\bar{u} = 0$ satisfying both BS.
To have a solution necessary condition

$$\langle h, \bar{u}_i \rangle = 0 \quad , \; i \leq 2 \quad (56)$$

If (54) & (56) are satisfied. there exist solns (otherwise there exist no solutions). To construct such solution we modify the Green's function equation

$$L_x^{+} \bar{g} = \frac{\delta(x-y)}{\omega} - \sum_{i=1}^{2} \bar{v}_i(x)\, \bar{v}_i(y) \quad (57)$$

$$L_x \bar{G} = \frac{\delta(x-y)}{\omega} - \sum \bar{u}_i(x)\, u_i(y) \quad (58)$$

The reason we add $\sum_{i=1} \bar{v}_i(x) v_i(x)$ terms in the Green's funct equation is clear.

Remark: If $\bar{v}_i$'s are the solutions of the $L^{+} v_i = 0$ satisfy both ad $B_i = 0$ then we can use the Green's Identity for the pair

(G, v_i) we get

$$\int \bar{v}_i L_x G \, w dx = \int G(L_x^{+} \bar{v}_i) \, w dx = 0$$

$\bar{v}_i = 0$ but we said v_i 's are non trivial.

Hence we modify the equation for G. (57)

$$L_x \, G'(x,y) = \frac{\delta(x-y)}{w} - \sum_i \bar{v}_i(x) \bar{v}_i(y)$$ v_i''s orthonormal

$\Rightarrow$ The solutions obtained this way will not be unique. To obtain unique solns we impose

$$\langle u_i | G' \rangle = 0 \qquad i \leq 2$$
$$\langle v_i | g' \rangle = 0 \qquad i \leq 2.$$

Use (v, G')

$$\int w \bar{v} \left[\frac{\delta}{w} - \sum \right] dx = \int w G' \bar{h} \, dx$$

$$\bar{v}(y) - \int w \, \bar{v} \sum \bar{v}_i(x) \, v_i(y) = \int w \, G' h \, dx$$

$$\bar{v}(y) + c_i \bar{v}_i(y) = \int w \, G' h \, dx$$

generalized Green's funct (gf 13')

$$L_x\, G'(x,y) = \frac{\delta(x-y)}{w} - \sum \bar{v}_i(x)\, v_i(y)$$

$$L_x^{\dagger}\, g'(x,y) = \frac{\delta(x-y)}{w} - \sum \bar{u}_i(x)\, u_i(y)$$

<u>with conditions</u> (to remove the arbitrariness in G)

$$\langle u_i | G' \rangle = 0 \qquad i \le 2$$

$$\langle v_i | g' \rangle = 0 \qquad i \le 2$$

(U, g)

$$\int_a^b w\left[\bar{g}'(x,y)\, L_x u(x) - u(x)\left(\overline{L_x^{\dagger} g'}\right)\right] dx$$

$$\int_a^b w\, \bar{g}'(x,y)\, f(x)\, dx - \int_a^b w\, u(x)\left[\frac{\delta(x-y)}{w} - \sum \bar{u}_i(x)\, u_i(y)\right]$$

$$- u(y) + \sum_{i} \langle u(x) | u_i(x) \rangle\, u_i(y) + \int_a^b w(x)\, f(x)\, \bar{g}'(x,y)\, dx = 0$$

$$u(y) = \sum_i \langle \bar{u}_i(x) | u(x) \rangle\, u_i(y) + \int_a^b w(x)\, f(x)\, \bar{g}'(x,y)\, dx$$

$$\underline{u(x) = \sum \langle \bar{u}_i | u \rangle\, u_i(x) + \int_a^b w(x)\, f(x)\, G'(x,y)\, dx}$$

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Example: $\frac{d^2u}{dx^2} = f(x) \qquad -a \leq x \leq a$

BCs: $u(a) = u(-a)$, $u'(a) = u'(-a)$.

$u_1 = 1$, $u_2 = x$.

$u_1 = 1$ satisfies the both boundary conditions.

$\bar{u}_1 = \frac{1}{\sqrt{2a}} \qquad \langle \bar{u}_1, f \rangle = 0 \Rightarrow \int_{-a}^{a} f(x)\,dx = 0$

$$\frac{d^2G}{dx^2} = \delta(x-y) - \frac{1}{2a}.$$

$$G(x,y) = \begin{cases} -x^2/4a + \frac{y-a}{2a}x + \alpha & x \leq y \\ -\frac{x^2}{4a} + \frac{y+a}{2a}x + \alpha - y & x \geq y. \end{cases}$$

$\left[u(x) = \alpha + \frac{x}{2a}\int_{-a}^{a} y f(y)\,dy + \int_{a}^{x} (x-y) f(y)\,dy. \right]$

In the general case one can write

$$u(x) = \int_a^b w(y) f(y) \bar{g}'(y,x)\,dy$$

but we can always add $\bar{u}_i$ to this equat

$$u(y) = \int_{-a}^{y} \frac{y-a}{2a} x f(x)\,dx + \int_{y}^{a} \left[\frac{y}{2a} + \frac{1}{2}\right] - y \quad + \alpha$$

$$u'(y) = \frac{1}{2a}\int_{-a}^{y} x f(x)\,dx + \int_{y}^{a} \left(\frac{x}{2a} - 1\right) f(x)\,dx$$

Generalized Green's Function

Let

$$Lu = f \qquad x \in (a,b)$$

$$B_1(u) = 0, \quad B_2(u) = 0$$

$L^{\dagger}_x = L_x$

i) Let u_1 and u_2 be solutions of the homogeneous eqn. $Lu = 0$

ii) Let u_1' and u_2' be orthonormalized solutions of the homogeneous eqn satisfying both BCs

$$\langle u_i', u_j' \rangle = 0 \quad , \quad L_x u_i' = 0$$

1) $L_x G(x,y) = \dfrac{\delta(x-y)}{w} - \sum_{i=1}^{2} \bar{u}_i'(x)\, u_i'(y)$

2) $u(x) = \sum_{i=1}^{2} c_i u_i'(x) + \int_a^b w(y)\, G(x,y)\, f(y)\, dy$

where c_i's are constants

3) $\langle u_i, f \rangle = 0 \qquad i = 1,2$

4) $\langle u_i, G(x,y) \rangle = 0 \qquad i = 1,2$

5) $G(x,y) = \begin{cases} u_p + c_1 u_1 + c_2 u_2 & x \le y \\ u_p + d_1 u_1 + d_2 u_2 & x \ge y \end{cases}$

$$L_x G = - \sum_{i=1}^{2} \hat{u}_i^*(x)\, \hat{u}_i(y)$$

a) $B_1(G) = 0, \quad B_2(G) = 0$

b) $G_+ = G_-$

c) $G_+' - G_-' = \dfrac{1}{aw}$

Theorem: Let $L_x u = f(x)$, $x \in (a,b)$ with boundary conditions $B_1(u) = 0$, $B_2(u) = 0$. Let there exist ~~2~~ nontrivial solutions of the homogeneous equation satisfying both boundary conditions. ~~If~~ If $\langle \hat{u}_i, f \rangle \neq 0$ then there exist no solution of this Boundary Value Problem. If $\langle \hat{u}_i, f \rangle = 0$ there exist infinitely many solutions

$$u(x) = c_1 \hat{u}_1(x) + c_2 \hat{u}_2(x) + \int_a^b w(y) G(x,y) f(y)\, dy$$

where $G(x,y)$ is the Green's function found from (5).

Example: $u'' = f \qquad x \in (-a,a)$

$$B_1(u) = u(-a) - u(a) = 0$$

$$B_2(u) = u'(-a) - u'(a) = 0$$

$$u_1 = 1, \quad u_2 = x$$

u_1 satisfies the both BCs

$$u_1' = \frac{1}{\sqrt{2a}} \qquad L_x G = \frac{\delta(x-y)}{w} - \frac{1}{2a}$$

$$L_x G = -\frac{1}{2a} \qquad x \neq y$$

$$u_p = -\frac{1}{4a} x^2$$

$$G(x,y) = \begin{cases} -\frac{1}{4a} x^2 + c_1 + c_2 x & x \leq y \\ -\frac{1}{4a} x^2 + d_1 + d_2 x & x \geq y \end{cases}$$

$$B_1(G) = -\cancel{\frac{a}{4}} + c_1 - c_2 a = -\cancel{\frac{a}{4}} + d_1 + d_2 a$$

$$c_1 - d_1 = (d_2 + c_2)\, a$$

$$B_2(G): -\frac{1}{2} + d_2 - c_2 + \frac{1}{2} = 0$$

$$\cancel{c_2 - d_2 = 1.} \quad d_2 - c_2 = 1$$

$G_+ = G_-$

$$c_1 + c_2 y = d_1 + d_2 y$$

$$c_1 - d_1 = (d_2 - c_2)\, y$$

jump: $d_2 - \cancel{\frac{y}{4a}} - c_2 + \cancel{\frac{y}{4a}} = 1 \qquad \underline{d_2 - c_2 = 1}$

$$c_1 - d_1 = y, \quad d_2 - c_2 = 1$$

$$(d_2 + c_2)\, a = y$$

$$d_2 = \frac{1}{2}(1 + y/a), \quad c_2 = \frac{1}{2}\left(\frac{y}{a} - 1\right), \quad c_1 = d_1 + y$$

$$G'(x,y) = \begin{cases} -\dfrac{x^2}{4a} + \dfrac{1}{2}\left(\dfrac{y}{a} - 1\right) x + d_1 + y \\ -\dfrac{x^2}{4a} + \dfrac{1}{2}\left(\dfrac{y}{a} + 1\right) x + d_1 \end{cases}$$

d_1 is arbitrary.
it can be found from $\langle \hat{u}_1, G' \rangle = $

$$u(x) = C\,\hat{u}_1(x) + \int_{-a}^{a} G'(x,y)\,f(y)\,dy$$

$$= \frac{C}{\sqrt{2a}} + \int_{-a}^{x} \left[-\frac{x^2}{4a} + \frac{1}{2}\left(\frac{y}{a}+1\right)x + d_1 \right] f(y)\,dy$$

$$+ \int_{x}^{a} \left[-\frac{x^3}{4a} + \frac{1}{2}\left(\frac{y}{a}-1\right)x + d_1 + y \right] f(y)\,dy$$

since $\langle \hat{u}_1, f \rangle = 0 \Rightarrow \int_{-a}^{a} f(y)\,dy = 0$

if $\int_{-a}^{a} f(y)\,dy \neq 0$ there exist no solution. if $\int_{-a}^{a} f(y)\,dy = 0$ then there exist infinitely many solution

$$u(x) = C_0 + \frac{x}{2}\int_{-a}^{x} \left(\frac{y}{a}+1\right) f(y)\,dy$$

$$+ \frac{x}{2}\int_{x}^{a} \left(\frac{y}{a}-1\right) f(y)\,dy + \int_{x}^{a} y f(y)\,dy$$

C_0 is any arbitrary constant

$$u(-a) = C_0 - \frac{a}{2}\int_{-a}^{a} \left(\frac{y}{a}-1\right) f(y)\,dy + \int_{-a}^{a} y f(y)\,dy$$

$$u(a) = C_0 + \frac{a}{2}\int_{-a}^{a} \left(\frac{y}{a}+1\right) f(y)\,dy \quad u(-a) = u(a)$$

$$u' = \frac{1}{2}\int_{-a}^{x} \left(\frac{y}{a}+1\right) f(y)\,dy + \frac{1}{2}\int_{x}^{a} \left(\frac{y}{a}-1\right) f(y)\,dy \; \underline{- \cancel{x}\, f(x)}$$

$$+ \frac{x}{2}\left(\cancel{\frac{x}{a}} + \underline{1}\right) f(x) - \frac{x}{2}\left(\cancel{\frac{x}{a}} - \underline{1}\right) f(x)$$

$$= \frac{1}{2}\int_{-a}^{x} \left(\frac{y}{a}+1\right) f(y)\,dy + \frac{1}{2}\int_{x}^{a} \left(\frac{y}{a}-1\right) f(y)\,dy$$

$$u'(-a) = + \frac{1}{2a}\int_{-a}^{a} y\, f(y)\,dy$$

$$u'(-a) = u(a) \checkmark$$

$$u(a) = \frac{1}{2a}\int_{-a}^{a} y\, f(y)\,dy$$

$$u'' = \frac{1}{2}\left(\cancel{\frac{x}{a}} + 1\right) f(x) - \frac{1}{2}\left(\cancel{\frac{x}{a}} - 1\right) f(x) = f(x) \checkmark$$

with inhomogeneous boundary conditions

$$B_1(u) = \sigma_1 \quad , \quad B_2(u) = \sigma_2$$

we find the Green's function as if the BCs were homogeneous. Then using the Green's identity we get

$$u(y) = \int_a^b dx\, w(x)\, G(y,x)\, f(x) \;+\; \underbrace{\mathcal{P}\left[u(x), G(y,x)\right]\Big|_{x=a}^{x=b}}_{\text{surface term}}$$

In particular, for a second order DE we had the surface term is

$$p(x)\left[G(y,x)\frac{du}{dx} - u(x)\frac{\partial}{\partial x} G(y,x) \right]_{x=a}^{x=b}$$

Example: $u'' = f(x) \qquad x \in [0,a]$

$$u(0) = \sigma_1 \quad , \quad u(a) = \sigma_2$$

$$G(x,y) = \begin{cases} \dfrac{y-a}{a}\, x & x \le y \\[2mm] \dfrac{y(x-a)}{a} & x \geq y \end{cases}$$

$$G(x,0) = 0 \quad , \quad G(x,a) = 0$$

$$\left.\frac{\partial G}{\partial x}\right|_{x=0} = \frac{y-a}{a} \quad , \quad \left.\frac{\partial G}{\partial x}\right|_{x=a} = \frac{y}{a}$$

$$u(y) = u_0(y) + \sigma_2 \frac{y}{a} - \sigma_1 \frac{(y-a)}{a}$$

$$u(y) = u_0(y) + \frac{\sigma_2 - \sigma_1}{a}\, y + \sigma_1$$

An easy way:

$$u = u_0 + a u_1 + b u_2$$

where a, b found fr

$$a B_1(u_1) + b B_1(u_2) = \sigma_1$$
$$a B_2(u_1) + b B_2(u_2) = \sigma_2$$

$$\alpha + \beta(0) = \sigma_1$$

$$\sigma_1 + \beta a = \sigma_2 \Rightarrow \beta = \frac{\sigma_2 - \sigma_1}{a}$$

$$u = u_0 + \sigma_1 + \frac{\sigma_2 - \sigma_1}{a} x$$

Green's functions and The Sturm-Liouville Problem:

We shall consider a Hermitian differential operator L, $L^{\dagger} = L$.

The study of the "eigenvalue" problem

$$L|u_\lambda\rangle = \lambda |u_\lambda\rangle$$

"Sturm Lioville problem. Existence of the inverse of L, the Green's function, guarantees that $\lambda \neq 0$.

- L has an infinity number of eigenvalue
- for a given eigenvalue there are finite number of eigenvector (eigen functions)

$$\lambda_n \text{ --- } u_n^{(k)} \qquad \text{multiplicity } k$$

- The eigenvectors of L span the space $L^2_w(a,b)$. Any vector $|f\rangle \in L^2_w(a,b)$ can be expanded in a series (the index k distinguishes between different eigenvectors corresponding to the same eigenvalue)

$$|f\rangle = \sum_{\lambda, k} \langle k, u_\lambda | f\rangle \, |u_\lambda, k\rangle$$

uniform convergence may ~~be~~ require some additional conditions on f.

(i) Eigenvalues are real
(ii) Eigenvector for different eigenvalues are orthogonal.

Eigenfunction expansion of Green's Functions

let $L = L^{\dagger}$ and take the equation

$$(L_x - \ell)\, u(x) = f(x), \qquad a \le x \le b$$

with some BCs. The solution of the homogeneous equation $(L_x - \ell) u = 0$ may be quite difficult. Hence we shall express the solution in terms of the eigenfunctions of L

$$L_x\, u_n^{(k)}(x) = \lambda_n\, u_n^{(k)}(x).$$

The Green's function

$$(L_x - \ell) G_\ell(x,y) = \frac{\delta(x-y)}{\omega}$$

let

$$G_\ell(x,y) = \sum_{n,k}{}' a_n^{(k)}(y)\, u_n^{(k)}(x).$$

Using the Green's identity $(G, \overline{u_n^k})$ pair.

$$\int \omega \left[G(x,y)(L_x - \ell)\overline{u_n^{(k)}}(x) - \overline{u_n^{(k)}}(x)\,(L_x - \ell) G \right] dx = 0$$

$$\int \omega \sum{}' a_m^{(l)}(y)\, u_m^{(l)}(x)\, \overline{u_n^{(k)}}(x)(\lambda_n^k - \ell)$$

$$- \overline{u}_n^{(k)}(y) = 0$$

$$a_n^{(k)}(y) = \frac{\overline{u}_n^{(k)}(y)}{\lambda_n^k - \ell}$$

hence

$$G_\ell(x,y) = \sum_{n,k} \frac{\bar{u}_n^{(k)}(y)\, u_n^{(k)}(x)}{\lambda_n - \ell}$$

<u>Remark</u>: $\sum' \bar{u}_n^{(k)}(y)\, u_n^{(k)}(x) = \delta(x-y)/\omega$

or, The sequence

$$h_N(x) = \omega \sum^{N}{}' \bar{u}_n^{k}(y)\, u_n^{(k)}(x)$$

$$\lim_{N\to\infty} \int_a^b \omega\, h_N\, g(x)\, dx = \int_a^b \delta(x-y)\, g(x)\, dx = g(y)$$

for all "good" functions

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Remark: We assumed that $\ell \neq \lambda_n$. If $\ell = \lambda_m$ for some "m" we exclude m from the sum.

<u>soln</u>: $\ell \neq \lambda_m$

$$u(x) = \int \omega f(x)\, G\, dx = \sum' \frac{\langle f, u_n^{(k)}\rangle\, u_n^{(k)}(x)}{\lambda_n - \ell}$$

$\ell \neq \lambda_m$

$$u(x) = C\, u_m(x) + \sum_{n \neq m} \frac{\langle f, u_n\rangle\, u_n^{(k)}}{\lambda_n - \ell}$$

C is a constant and $\langle u_m, f\rangle = 0$

(u, u_m) pairing al

Example: $u'' + l u = f$, $u(0) = u(a) = 0$

a) Solution of the homogeneous equation $u'' + lu = 0$

$\sin\sqrt{l}\, x$, $\cos\sqrt{l}\, x$, $x \in [0, a]$

$\Rightarrow$ Green's function

$$G_l(x,y) = \frac{1}{\sqrt{l}\sin(\sqrt{l}a)}\Big[\sin(\sqrt{l}x)\sin(\sqrt{l}(a-y))\,\Theta(y-x) + \sin(\sqrt{l}(a-x))\sin(\sqrt{l}y)\,\Theta(x-y)\Big]$$

$\sqrt{l}\, a \neq n\pi$

b) Eigen functions : $u'' = -\lambda_n u$

$$\lambda_n = \frac{n^2\pi^2}{a^2} \qquad n = 0, 1, 2.$$

$$u_n = \sqrt{\frac{2}{a}}\sin\frac{n\pi}{a}x$$

The above Green's function can be written as

$$G = \frac{2}{a}\sum_{n=0}\left(\sin\frac{n\pi y}{a}\right)\left(\sin\frac{n\pi x}{a}\right)\frac{1}{-\lambda_n + l}$$

~~$u'' = -\lambda_n u$~~

~~$u'' + \lambda_n u = 0$~~

$$G'' = -\frac{2}{a}\sum \lambda_n \sin\sin\frac{1}{-\lambda_n + l}$$

$$G'' + lG = -\frac{2}{a}\sum(\lambda_n - l)\sin\sin\frac{1}{\lambda_n - l}$$

$$= +\delta(x-y).$$

SET 5

MATH 543: GREEN'S FUNCTIONS

(References: DK, Sadri Hassan and Hildebrandt)

Generalized Green's Identity

Let L_x be a differential operator and u , v are some functions in $L^2_w(a, b)$ then the *adjoint operator* $L^\dagger$ is defined through the *Lagrange identity*

$$w[\bar{v} L_x u - u\,\overline{\{L_x^\dagger v\}}] = \frac{d}{dx} Q(u, \bar{v}), \tag{1}$$

where $w > 0$ is the weight function and $Q(u, \bar{v})$ has in general the following form

$$Q(u, \bar{v}) = A u \bar{v} + B \bar{v} \frac{du}{dx} + C u \frac{d\bar{v}}{dx} + D \frac{du}{dx} \frac{d\bar{v}}{dx}, \tag{2}$$

Integrating the above identity over the interval $[a, b]$ we get the *generalized Green's identity*

$$\int_a^b w(x)\, [\bar{v}\, L_x u - u\,\overline{\{L_x^\dagger\, v\}}] dx = Q(u(b), \bar{v}(b)) - Q(u(a), \bar{v}(a)), \tag{3}$$

The right hand side of this equation is called the *surface term.*

Adjoint Boundary Conditions:

Let the function u satisfy the boundary conditions

$$\begin{aligned} B_1(u) &= \alpha_1 u(a) + \beta_1 u'(a) + \gamma_1 u(b) + \delta_1 u'(b) = 0, && (4) \\ B_2(u) &= \alpha_2 u(a) + \beta_2 u'(a) + \gamma_2 u(b) + \delta_2 u'(b) = 0, && (5) \end{aligned}$$

Here $\alpha_i, \beta_i, \gamma_i$, and δ_i are some given constants. If in the above *generalized Green's identity* (3) the function u satisfies the above boundary conditions (4) and (5) then the function v satisfies the *adjoint boundary conditions* when the *surface term vanishes.* Hence for future purposes we can define the *Green's identity*

$$\int_a^b w[\bar{v}\, L_x u - u\, \overline{\{L_x^\dagger\, v\}}]dx = 0, \tag{6}$$

for the functions u satisfying the boundary conditions (4) and (5) and v satisfying the *adjoint boundary conditions*

Self Adjoint Operators:

We shall be interested in second order differential operators

$$L_x u(x) = a(x)\frac{d^2u}{dx^2} + b(x)\frac{du}{dx} + c(x)u, \tag{7}$$

where the coefficients a, b and c are some functions of x. Here we assume that $Re(a) > 0$. It is easy to show that the adjoint operator is given by

$$L^\dagger v = \frac{d^2}{dx^2}(\bar{a}\, v) - \frac{d}{dx}(\bar{b}v) + \bar{c}v, \tag{8}$$

and the function Q in the surface term is found as

$$Q(u, \bar{v}) = a\bar{v}\frac{du}{dx} - u\frac{d}{dx}(a\bar{v}) + bu\bar{v}, \tag{9}$$

If the coefficient functions are real and

$$b(x) = \frac{da(x)}{dx}, \tag{10}$$

Then $L_x^\dagger = L$, becomes a self-adjoint operator and the Q term becomes.

$$Q(u, \bar{v}) = a(\bar{v}\frac{du}{dx} - u\frac{d\bar{v}}{dx}), \tag{11}$$

With a suitable weight factor all second order differential operators are self-adjoint

Theorem 1. *Every linear second order differential operator with real coefficients is self adjoint provided the weight function $w(x)$ is chosen properly*

$$w(x) = \frac{p}{a}, \quad \frac{p'}{p} = \frac{b(x)}{a(x)}, \tag{12}$$

where the differential operator takes the form

$$L_x u = \frac{1}{w}\left(\frac{d}{dx}\, p(x) \frac{du}{dx}\right) + cu, \tag{13}$$

and the function Q is given by

$$Q(u, \bar{v}) = p(x)\,(\bar{v}\frac{du}{dx} - u\frac{d\bar{v}}{dx}), \tag{14}$$

Problems:

1. Prove that the following boundary conditions on $u(x)$, L_x defines a Hermitian differential operator L

(i) $u(a) = u(b) = 0$ (Dirichlet conditons),

(ii) $\frac{du}{dx}(a) = \frac{du}{dx}(b)$, (Neumann conditions),

(iii) $\alpha\, u(a) - \frac{du}{dx}(a) = \beta u(b) - \frac{du}{dx}(b)$, ($\alpha, \beta$ real) (general unmixed conditions),

(iv) $u(a) = u(b)$ and $\frac{du}{dx}(a) = \frac{du}{dx}(b)$, (periodic conditions).

2. Determine the formal adjoint of each of the operators in (a) through (d) below (i) os a differential operator, and (ii) as an operator, that is, including the boundary conditions (Determine the functions spaces of L and $L^{\dagger}$). Which operators are formally self-adjoint? Which operators are self- adjoint?

(a) $L_x = \frac{d^2}{dx^2} + 1$ in $[0, 1]$ with $u(0) = u(1) = 0$,

(b) $L_x = \frac{d^2}{dx^2}$ in $[0, 1]$ with $u(0) = u'(0) = 0$,

(c) $L_x = \frac{d}{dx}$ in $[0, \infty)$ with $u(0) = 0$,

(d) $L_x = \frac{d^3}{dx^3} - \sin x \frac{d}{dx} + 3$ in $[0, \pi]$ with $u(0) = u'(0) = 0$ and $u''(0) - 4u(\pi) = 0$

Boundary value problems

Let u satisfy the inhomogenous differential equation with homogenous boundary conditions

$$L_x u(x) = f(x), \quad B_1(u) = 0, \quad B_2(u) = 0, \tag{15}$$

where $f(x)$ is any integrable function. The function v satisfying the adjoint differential and the homogenous adjoint boundary conditions

$$L_x^\dagger v(x) = h(x), \quad adB_1(v) = 0, \quad adB_2(v) = 0, \tag{16}$$

Green's Functions:
Green's functions $G(x, y)$ and $g(x, y)$ of the above boundary value problems are, respectively given by

$$L_x G(x, y) = \frac{\delta(x-y)}{w(x)}, \tag{17}$$

$$L_x^\dagger g(x, y) = \frac{\delta(x-y)}{w(x)}, \tag{18}$$

using the Gren's identity for the function $u, v, G,$ and g pairwise (u, v), (u, g), (v, G) and (G, g) we obtain the following important identities

$$u(x) = \int_a^b w(y)\bar{g}(y, x)\, f(y)dy, , \tag{19}$$

$$v(x) = \int_a^b w(y)\bar{G}(y, x)\, h(y)dy, \tag{20}$$

and

$$< f, g > = < G, h >, \quad G(x, y) = \bar{g}(y, x), \tag{21}$$

Since the solutions are given in (19) and (20), then the boundary value problems (15) and (16) reduces to the determination of the Green's functions G and g

Second order operators and Green's functions:
For a second order operators with real coefficients we have

$$\begin{aligned}
\frac{d}{dx}(p(x)\frac{du}{dx}) + cwu &= f(x), \quad B_1(u) = 0, \quad B_2(u) = 0, & (22)\\
\frac{d}{dx}(p(x)\frac{dv}{dx}) + cwv &= h(x), \quad adB_1(v) = 0, \quad adB_2(v) = 0, & (23)
\end{aligned}$$

and the Green's functions satisfy the following equations

$$\begin{aligned}
\frac{d}{dx}(p(x)\frac{dG(x,y)}{dx}) + cwG(x,y) &= \delta(x-y), & (24)\\
\frac{d}{dx}(p(x)\frac{dg(x,y)}{dx}) + cwg(x,y) &= \delta(x-y), & (25)
\end{aligned}$$

Since the Greens's functions satisfy the homogenous equation ($L_x G(x,y) = 0$) for $x \neq y$ the we have

$$G(x,y) = \begin{cases} c_1 u_1(x) + c_2 u_2(x) & a \leq x < y \\ d_1 u_1(x) + d_2 u_2(x) & x < y \leq b \end{cases} \tag{26}$$

where u_1 and u_2 are the independent solutions of the homogenous equation $L_x u = 0$ and c_1, c_2, d_1, d_2 are constants (wrt x) to be determined by the following conditions on G. First two are the homogenous boundary conditions and the rest two are the continuity of G and jump condition for the derivative of G

$$\begin{aligned}
B_1(G) = 0, \quad B_2(G) &= 0, & (27)\\
G(x,y)|_{x=y_+} &= G(x,y)|_{x=y_+}, & (28)\\
\frac{dG(x,y)}{dx}|_{x=y_-} - \frac{dG(x,y)}{dx}|_{x=y_-} &= \frac{1}{p(y)}, & (29)
\end{aligned}$$

We have the following important theorem.

Theorem 2. *Consider the boundary value problem $L_x u = f(x)$ with the homogenous boundary conditions $B_1(u) = 0$ and $B_2(u) = 0$. Here we assume* *that $L_x$is a second order differential operator. Provided that the homogenous equation has no nontrivial solution satisfying the above boundary conditions the Green's function associated with the boundary value problem exist and unique. The solution is given by*

$$u(x) = \int_a^b w(y)\, G(x,y)\, f(y)dy \tag{30}$$

Problems:

3. Let $B_1(u) = 0$ and $B_2(u) = 0$ are defined at $x = a$ and $x = b$ respectively. Solve the Green's function defined in (26).

Solution: The continuity conditions (28) and (29) are solved easily (see DK page 281)

$$G(x,y) = \begin{cases} \frac{U_<(x)U_>(y)}{p(y)W} & a \leq x < y \\ \frac{U_<(y)U_>(x)}{p(y)W} & y < x \leq b \end{cases} \tag{31}$$

where $W = U_< U_>' - U_> U_<'$ is the Wronksian of $U_<(x) = u_1(x) - \alpha\, u_2(x)$ and $U_>(x) = u_1(x) - \beta u_2(x)$. Here u_1 and u_2 are independent solutions of the homogenous DE (the fundamental set). Prove that $p(x)W(U_<, U_>)$ is constant for all x. The boundary conditions (27) mus also be satisfied, hence

$$B_1(U_<) = B_1(u_1) - \alpha B_1(u_2) = 0,$$
$$B_2(U_>) = B_2(u_1) - \beta B_2(u_2) = 0$$

Hence α and β are determined. As we discussed above the product pW is assumed to be a nonvanishing constant. If it is zero, then $U_<(x) = \sigma U_>(x)$, where σ is an arbitrary constant. Using the definitions of $U_<$ and $U_>$ and α and β found above this relation gives ,

$$B_1(u_1 - \rho u_2) = 0, \quad B_2(u_1 - \rho u_2) = 0$$

We obtain this by letting $\sigma = \rho \frac{B_2(u_2)}{B_2(u_1)}$, where ρ is another constant. The above relation implies the existence of a nontrivial solution $u_1 - \rho u_2$ of the homogenous DE satisfying the homogenous boundary conditions, but this is contradiction with our assumption.

4. Solve the Green's function when both boundary conditions are given at the same point $x = a$. As an example take for instance $u(a) = 0$ and $u'(a) = 0$.

Boundary value problems with inhomogenous boundary conditions: Let $B_1(u) = \gamma_1$ and $B_2(u) = \gamma_2$, where γ_1 and γ_2 are some given constants. In this case the construction of the Gree's function is exactly similar as above. We determine the Green's function as if the problem is with homogenous boundary conditions. Here the expression for (30) u changes. To obtain the expression for u we use the Green's identity, but in this case since the boundary conditions are homogenous we cannot assume that the surface term is zero. For the operator L_x given in (13) the surface term is given in (14). Now using the generalized Green's identity we get

$$u(x) = \int_a^b w(y)G(x,y)f(y)dy + \{p(y)[u(y)\frac{\partial G(x,y)}{\partial y} - G(x,y)\frac{du}{dy}]\}|_{y=a}^{y=b}, \quad (32)$$

here we use $B_1(u) = \gamma_1$ and $B_2(u) = \gamma_2$. In the above expression (32) we have $u(a), u(b)$ and $u'(a), u'(b)$ terms in the surface term. Two of them are inserted from the inhomogenous boundary conditions $B_1(u) = \gamma_1$ and $B_2(u) = \gamma_2$ and the rest dissapears from the expressions due to adjoint homogenous boundary conditions (for self dual operators due to the same homogenous boundary conditions)

Problems:

5. In problem 3 , if $B_1(u) = \gamma_1$ and $B_2(u) = \gamma_2$ then the complete solution will be

$$u(x) = \int_a^b w(y)\, G(x,y)\, f(y)dy + \frac{U_<(x)}{U_<(b}\gamma_2 + \frac{U_>(x)}{U_>(a)}\gamma_1$$

6. Solve $u'' + u = f(x)$ with $u(0) = u'(0) = 0$.

7. Find the Green's function for $L_x = \frac{d^2}{dx^2} + k^2$ with $u(0) = u(a) = 0$.

8. Find the Green's function for $L_x = \frac{d^2}{dx^2} - k^2$ with $u(\infty) = u(-\infty) = 0$.

9. Find the Geen's function for $L_x = \frac{d}{dx}\, x \frac{d}{dx}$ given the condition that $G(x, y)$ is finite at $x = 0$ and vanishes at $x = 1$.

10. Problem 5 can be generalized further. Let $L_x u = f$ with boundary condition $B_1(u) = \gamma_1$ and $B_2(u) = \gamma_2$. Here L_x is a second order differential operator with real coefficients. Prove that the solution is given by

$$u(x) = \int_a^b w(y)\, G(x, y)\, f(y) dy + \frac{1}{\Delta}[B_2(u_2)\, u_1(x) - B_2(u_1) u_2(x)]\, \gamma_1 + \frac{1}{\Delta}[B_1(u_1) u_2(x) - B_1(u_2) u_1(x)]\, \gamma_2 \quad (33)$$

where $\Delta = B_1(u_1) B_2(u_2) - B_1(u_2) B_2(u_1)$ and the Green's function is found as if the boundary conditions are homogenous. Discuss the case where $\Delta = 0$.

When there exist a nontrivial solution of the homogenous equation satisfying the homogenous boundary conditions:

Let v_i be the nontrivial solutions of the homogenous adjoin DE satisfying adjoint boundary conditions . Hence using the pair (v_i, G) we obtain $v_i = 0$, from (20) by letting $h = 0$ (v satisfies the homogenous equation). To resolve this we must modify the differential equations for G and g. They are

$$L_x G(x, y) = \frac{\delta(x-y)}{w(x)} - \sum_{k=1} \bar{v}_i(x)\, v_i(y), \quad (34)$$

$$L_x^\dagger g(x, y) = \frac{\delta(x-y)}{w(x)} - \sum_{k=1}^{N} \bar{u}_i(x)\, u_i(y) \quad (35)$$

where $u_i, (i \leq 2)$, $v_i, (i \leq 2)$ are orthonormalized solutions of homogenous $L_x u = 0$ and $L_x^\dagger v = 0$ satisfying homogenous boundary and adjoint boundary conditions respectively. To restore the uniqueness of the solutions we have the conditions

$$< u_i, G >= 0, (i \leq 2) < v_i, g >= 0 (i \leq 2) \tag{36}$$

For the self adjoint operators and for the case there exists only one homogenous solution satisfying the homogenous boundary conditions we have

$$L_x G(x,y) = -\bar{e}(x)e(y), \quad x \neq y \tag{37}$$

$$\int_a^b G(x,y)\, e(x)dx = 0, \tag{38}$$

here e is normalized solution of the homogenous equation satisfying both homogenous boundary conditions.

See the example in DK page 284. For the solutions to exist we must have (from (21, since $h = 0$)

$$< f, e >= \int_a^b f(x)e(x)dx = 0$$

If this condition is satisfied the solution of boundary value problem is given by

$$u(x) = \alpha e(x) + \int_a^b w(y)G(x,y)\, f(y)dy$$

where $e(x)$ is any solution of the homogenous equation satisfying the homogenous boundary conditions. Hence we have the following theorem

Theorem 3. *Assume that there exists nontrivial solutions of the homogenous DE $Lu = 0$ satisfying the both boundary conditions $B_1(u) =$ and $B_2(u) = 0$ boundary. Then, either the solutions of the boundary value problem $Lu = f$ with $B_1(u) = 0$, $B_2(u) = 0$ do not exist or (if it exists) they are not unique.*

Problems:

11. The solution of the example in DK (page 284) is given as follows: Solution does not exist if $\int_{-a}^{a} f(y)dy \neq 0$. If the equality holds then there are infinitely many solutions of the boundary value problem.

$$u(x) = \alpha + \frac{x}{2a}\int_{-a}^{a} yf(y)dy + \int_{-a}^{x}(x-y)f(y)dy$$

where α is any constant.

12. Solve $u'' + u = f(x)$ with $u(0) = u(\pi) = 0$

13. Solve $u'' = f(x)$ with $u(0) = 0$ and $u'(1) = 1$

Eigenfunction expansion of the Green,s functions

Theorem 4. *Let L_x be second order differential operator and let $L_x u = f(x), x \in [a, b]$ with the boundary conditions $\cos\alpha u(a) - \sin\alpha u'(a) = 0$ and $\cos\beta u(b) + \sin\beta u'(b) = 0$. The we have the following: (i) the spectrum consists entirely of eigenvalues (ii) The eigenvalues are countable and can be listed in a sequence $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ with $lim_{n\to\infty}\lambda_n = 0$ (iii) the set of normalized eigenfunctions $\{u_i\}$ is an orthonormal basis for $L^2_w(a, b)$ (iv) For the equation $L_x u - \lambda u = f(x)$ exactly one of the following hold:*
(a) If λ is not an eigenvalue of L then the solution is unique for every $f \in L^2_w(a, b)$. This solution is given by

$$u(x) = \sum_{i=1}^{\infty} \frac{(u_i, f >}{\lambda_i - \lambda} u_i(x)$$

(b) If $\lambda = \lambda_j$ is an eigenvalue then

$$u(x) = Cu_j(x) + \sum_{i\neq j}^{\infty} \frac{< u_i, f >}{\lambda_i - \lambda} u_i(x)$$

provided $< f, u_j >= 0$ otherwise there is no solution. Here C is an arbitrary constant

Problems

14. Consider the Sturm-Liouville eigenvalue problem consisting of the differential equations $(L_x u + \lambda u) = 0$ and the boundary conditions $B_1(u) = 0$ and $B_2(u) = 0$

(i) $u'' + \lambda u = 0,\ \ u(0) = 0,\ \ \cos\beta\, u(1) + \sin\beta u'(1) = 0$

(ii) $u'' + \lambda u = 0, \quad u'(0) = 0, \quad \cos\beta\, u(1) + \sin\beta\, u'(1) = 0$
(iii) $(xu')' + \lambda x u = 0, \quad u(a) = u(b) = 0$

Higher dimensional Green's functions:
Let ∇^2 denote the Laplace operator in $n-$ dimensions. Then $\nabla^2 G(\mathrm{x}) = \delta(\mathrm{x})$ where $\mathrm{x} \in \mathrm{R}^n$ has solution (see Set 4, problem 8)

$$G(\mathrm{x}) = \begin{cases} c_n\, r^{2-n} & n \geq 3 \\ c_2 \ln r & n = 2 \end{cases}$$

where

$$\begin{aligned} c_n &= -\frac{1}{(n-2)A_n}, \quad n \geq 3 \\ c_2 &= \frac{1}{2\pi} \end{aligned}$$

Here A_n is the surface area of the unit sphere in n dimensions given by $A_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$

More problems about the Green's function technique:

1. Evaluate the Green's function and solutions for each of the following differential equations in the interval $[0, 1]$.

(a) $u'' - k^2 u = f, \quad u(0) - u'(0) = a, u(1) = b,$
(b) $u'' = f, \quad u(0) = u'(0) = 0,$
(c) $u'' + 6u' + 9u = 0, \quad u(0) = 0, u'(0) = 1,$
(d) $u'' + w^2 u = f$, for $x > 0 \quad u(0) = a, u'(0) = 1,$
(e) $u^4 = f, \quad u(0) = 0, y'(0) = 2u'(1), u(1) = a, u''(0) = 0$

2. (From Hildebrand *Advanced Calculus for Applications*)
Consider $u'' + u = f(x)$ with $u(0) = 0$ and $u(a) = 0$ with $\sin a \neq 0$
(a) Show that

$$u(x) = \int_0^x f(y) \sin(x - y) dy + c \sin x$$

where c is such that

$$c \sin a = -\int_0^a f(y) \sin(a - y) dy$$

Hence

$$u(x) = \frac{1}{\sin a} \left[\int_0^x f(y) \sin(x - y) \sin a dy + \int_0^a f(y) \sin(a - y) \sin x \, dy \right]$$

(b) Prove that

$$G(x, y) = \begin{cases} \frac{\sin y \sin(x-a)}{\sin a} & y \leq x \\ \frac{\sin(y-a) \sin x}{\sin a} & x \leq y \end{cases}$$

when $\sin a \neq 0$
(c) If $\sin a = 0$, show that the above equations has no solution unless $f(x)$ satisfies the condition

$$\int_0^a f(x) \sin(a - x) dx = 0$$

in which case there are infinitely many solutions , each of the form

$$u(x) = \int_0^x f(y)\,\sin(x-y)dy + C\sin x$$

where C is an arbitrary constant

3. Solve $u'' = f(x)$ with $u(0) = \alpha$ and $u'(a) = \beta$ and show that

$$u(x) = \alpha + \beta x + \int_0^a G(x,y)f(y)dy$$

with

$$G(x,y) = \begin{cases} -x & x \leq y \\ -y & x \geq y \end{cases}$$

4. Solve $u'' - u = f(x)$ with $u(-\infty) = 0$ and $u(\infty) = 0$ and show that

$$u(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|}\,f(y)dy$$

5. Solve $u'' - \frac{1}{x}u' = f(x)$ with $u(0) = 0$ and $u(1) = 0$ and show that

$$u(x) = \int_0^1 G(x,y)f(y)dy$$

with

$$G(x,y) = \begin{cases} -\frac{(1-y^2)x^2}{2y} & x \leq y \\ -\frac{y(1-x^2)}{2} & x \geq y \end{cases}$$